A Design Procedure for Maximizing Altitude Performance

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By

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Abstract

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Our interest in dynamic stability is centered on a desire to create a design procedure that will maximize altitude performance. Dynamic stability plays a principal role in determining how straight a rocket will fly. The main thrust of this work is to produce a practical "Top Down" design method and to validate the procedure. Any design procedure to maximize altitude must take into account the influences of wind, thrust and airframe misalignments that will degrade performance.

Early work on dynamic stability omitted the influence of the translational axis and resulted in some erroneous conclusions. Chief among them is that a rocket with a large longitudinal inertia will experience severe resonance. This is not the case, as we shall show that these rockets are well damped.

This R&D report will demonstrate that a design procedure may be developed to maximize altitude in the presence of wind, airframe and thrust errors. For a given set of specifications on wind, airframe and thrust errors, there is an optimum fin design that will maximize altitude performance. Simple approximations to the actual performance provide an easy means to estimate results and provide insight into the various effects. The analysis and design procedure results in the following important conclusions:

- There is an optimum fin size which will maximize altitude in the presence of combinations of thrust and wind errors.

- The launcher length and angle are critical factors in dealing with wind and thrust errors. There is an optimum launcher length. The error due to wind may be minimized by tilting the launcher with the wind by an amount equal to the estimated flight path angle, γ, caused by the wind.

- The optimum fin size due to the effects of wind and airframe errors is as small as possible, but, not allowing the peak angle of attack to exceed 15-20 degrees. Small fins also may have very small angles of attack to reach unstable pitching moments. Thus, making fins small to avoid wind errors may make the rocket unstable at a low wind speed. The launcher length should be selected to keep the peak angle of attack below 15-20 degrees.

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Background

Our interest in dynamic stability is centered on a desire to create a design procedure that will maximize altitude performance. Dynamic stability plays a principal role in determining how straight a rocket will fly. The main thrust of this work is to produce a practical “Top Down” design method and to validate the procedure. Any design procedure to maximize altitude must take into account the influences of wind, thrust and airframe misalignments that will degrade performance.

Early interest in the topic of dynamic stability was initiated by the publication of the now famous “Barrowman Method” [1] of the calculation of the static stability of a rocket. The primary focus of this landmark work is the calculation of the aerodynamic moments and center of pressure of the rocket and all its components. The method stands, to this day, as the method of choice for static stability calculations. Static stability is defined as the distance between the center of pressure and the center of gravity of the rocket. It is usually measured in fractions of the diameter or caliber.

Building on the foundations laid out for static stability by Barrowman, Mandell et.al. [2] created a massive body of knowledge about the theoretical aspects of dynamic stability. This equally powerful work is the present-day foundation for our understanding of model rocket dynamics. This work, as well as the National Association of Rocketry Technical Services (NARTS) technical bulletin on dynamic stability by Mandell [3], provided excellent reference material related to the rotational aspects of rocket stability. The main limitation of the Mandell material is the omission of the translational axis influence on the rotational dynamics. This omission can lead to some incorrect conclusions. Chief among them is that a rocket with a large longitudinal inertia will experience severe resonance. This is not correct, as we shall see later. We believe that previous work placed too much emphasis on the damping factor. In our work, we will completely ignore the damping aspects of dynamics. We further believe that any design procedure that attempts to “optimize” damping properties will degrade altitude performance while gaining little else.

Despite the depth and breadth of the Mandell work on dynamic stability, there is still some confusion about how to interpret the results. Micci [4], for example, erroneously states that a model rocket with high moment of inertia will be dynamically unstable even if it is statically stable. Perhaps this error finds its source in the omission of the translational effects on stability. Or perhaps the author believes if a rocket may be seen to “wiggle” it is “unstable.”

Definition of Stability

We would like to make clear our definition of stability. In engineering terms, the word “stable” means that the poles of a dynamical system lie in the “left half plane.” A stable system is one that will cause any disturbance to eventually dampen out. The modern control system [5] definition of stability, taken after Lyapunov’s work, is based on the notion that a system subject to a small disturbance will return to the equilibrium condition or will remain within a preassigned finite region of the equilibrium state. This means that a small oscillation about the equilibrium state is considered stable.

A good example to illustrate this concept is a pendulum. A vertical pendulum has two states, a stable state hanging down, and an unstable state balanced upward. When in the stable state, if disturbed, it will return to the equilibrium state. If disturbed in the unstable state, it will not return to the original state. Rockets are similar to pendulums. Their equations of motion are similar in that they both may be considered as "simple harmonic oscillators." Harmonic oscillators have equations of motion that are usually in the form of equation (1).

$$X = A\,Sin(\omega\, t)e^{k\,t} \tag{1}$$

The engineering definition of stability demands that k be zero or negative in order to be stable. If k is positive the magnitude of the oscillations will grow without bound. In practical situations, the oscillations will grow until some nonlinear effect, such as saturation, will cause k to become zero. In model rockets, the sign of k is determined by the static margin. If the Cp lies behind the Cg k is negative and the rocket is stable.

The word "dynamic" in the usual engineering sense means time varying. Technically the interpretation of the words "Dynamic Stability" would imply "time varying stability." This is not the usual meaning in the world of model rockets. Dynamic stability, in model rocket terms, means dynamic behavior, not stability. That is, dynamic stability refers to describing the "wiggling" motion of a rocket. Undergoing a damped oscillation is not a sign of instability.

Concept of Stability and Model Rockets

Model rockets can become "dynamically" unstable/stable if the Cp varies with time, speed or angle of attack. In previous work [6] we developed a simple model to extend the Barrowman method of Cp calculation to large angles of attack. This work focuses on the variation in Cp location with angle of attack, α. In most rockets, the Cp will move aft with increasing angles of attack. This is due to a change in the Cp location of the body. If the angle of attack becomes too large, the pitching moment, Cm, can become positive (unstable). This effect is shown in Figure 1 below. The point of positive pitching moment may be used as a design constraint so that stability may be assured throughout the flight profile. The designer must ensure that the peak angle of attack caused by wind and airframe errors or thrust misalignment fall short of the unstable point with some margin.

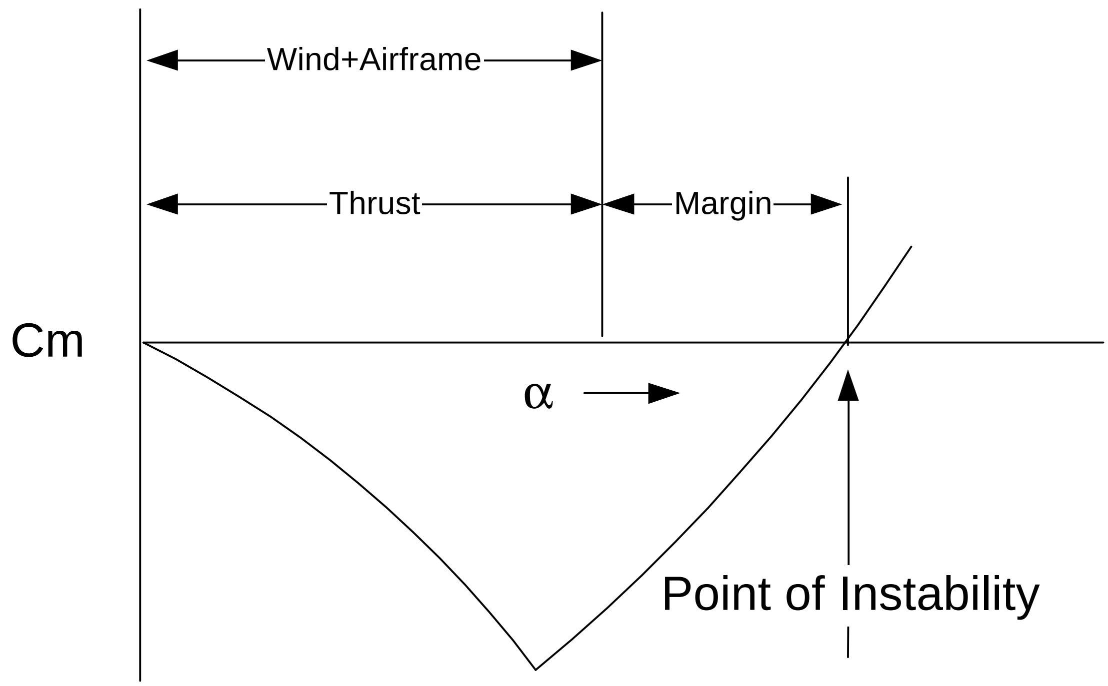


Figure 1 Pitching Moment

A model rocket with no fins is unstable by our definition. So are spears and javelins. If one throws a javelin it looks like it is stable. The Cp of a finless airframe usually lies forward of the Cg, making it unstable at zero angle of attack. The Cp will move aft with increasing angle of attack. For most slender bodies the Cp will reach the Cg at an angle of attack between 20 and 40 degrees. Just as the pendulum will reach a stable state when disturbed from the unstable state, a finless vehicle will reach a stable angle of attack. It is difficult to see this happen with a spear, but it happens. A javelin is best launched at about 25 degree angle of attack which corresponds to the stable point. If a rocket is flown without fins it <u>really</u> looks unstable! In rockets, a constant angle of attack means a constant turn radius, so it flies in a circle. A 40 degree stability point would make it render a right angle turn almost immediately.

In the case of maximizing altitude, it will be necessary for the rocket to be stable, so that it will fly straight up. The present design rules used by many modelers, such as a static margin of between 1 and 2, while being good advice, may not achieve optimum altitude performance. Our goal is to achieve maximum altitude with a guarantee that acceptable stability is available to deal with effects of dynamic disturbances.

Analysis

The thrust of our work is to fill the void left by previous authors for quantitative methods for estimating the actual dynamic performance of rocket designs. Previous work focused on the theory of dynamics and stopped short of numerical design procedures. We wish to find simple approximations to the dynamic performance so that a good sense for the parameters affecting performance may be obtained and are simple enough to be performed by hand calculation. That is a challenging goal when one considers how dynamic the environment really is. All aspects of the launch phase may vary considerably. The motor thrust can have large peaks, the accelerations are near maximum, velocity and position are changing, and the rocket might be rotating.

Despite this, some simplified assumptions may be created which will be of sufficient accuracy to enable a first-cut design. Design verification will be accomplished with more accurate design tools such as time domain simulations.

Approach

The complexity of the dynamics may be simplified considerably. We will consider a zero roll rate system so that we can neglect cross-coupling effects between pitch and yaw. We feel that this will be a worst-case scenario, since any roll will reduce the deviation from vertical flight. This enables us to reduce a 6 degrees of freedom (6DOF) problem to a maximum of a 3DOF (rotation [pitch], Y and X). We can further reduce the 3DOF to a 2DOF by making small angle approximations. The 2DOF will consist of rotation and X translation. Refer to the block diagram in Figure A1 in Appendix A to better visualize the 2DOF system. The Y axis may be estimated with simple approximations involving estimates of the averages of some of the dynamic parameters.

Next, we will validate the accuracy of these simple estimates using the 3DOF time domain simulation. The simulation will contain all of the full dynamics of the time varying parameters so we can determine how valid the assumptions are.

Finally we will compile a detailed step-by-step design procedure that will ensure maximum altitude, yet still be stable enough to deal with disturbances caused by wind, thrust, and airframe misalignment.

Vertical Axis

The ultimate goal is to achieve the maximum altitude (Y axis). The Y axis performance is also the easiest to approximate. The vertical situation is shown in Figure 2 below. A rocket subject to forces caused by wind, thrust and airframe misalignments will make a turn after clearing the launcher. We assume that these forces appear as step forces at the launch tip. The flight path will be deflected from vertical by an angle γ. This will cause a loss in altitude by two effects. First, the angle will cause a cosine loss in altitude. Second, the turn will consume energy by the added drag while making the turn. It should be obvious that a high altitude can not be achieved with a large flight path angle.

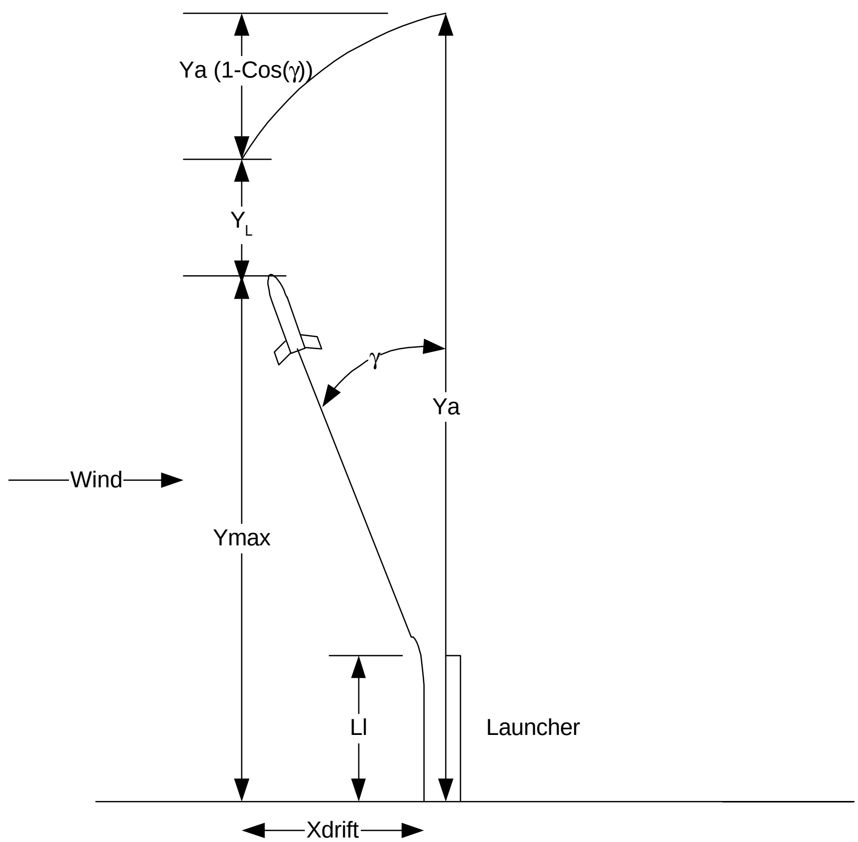


Figure 2 Vertical (Y Axis)

A full development of the Y axis approximations is contained in Appendix C. The maximum altitude simply becomes:

$$Y_{max} = Y_a Cos(\gamma) - Y_L \text{ ; Maximum altitude} \qquad (2)$$

Where;

$$Y_L = \frac{E_T}{m_C\ g} \text{ ; Turn loss altitude}$$

$$E_T = \frac{M_T\ V_{Tip}^2\ \alpha_{PEAK}}{L/D} \text{ ; Turn Loss Energy}$$

$$L/D \approx \frac{CN\alpha\,\alpha_{PEAK}}{Cdo + CN\alpha\,\alpha_{PEAK}^2} \text{ ; Lift to Drag ratio}$$

The formulas in Appendix C will allow the calculation of the estimate of Ya and will not be repeated here.

Our problem of dynamics is reduced to coming up with estimates for the flight path angle, γ (horizontal velocity), and the peak angle of attack during the turn, αpeak, in the presence of disturbances.

Rotation and Horizontal Axes

We would like to take this opportunity to broaden the generalization of the rotation axis by including the influence of the translation axis. The angle of attack, α, is related to the body angle, θ, and flight path angle, γ, as $\alpha=\theta-\gamma$. Previous work by Mandell omitted the influence of the translation axis by neglecting the flight path angle, γ. The consequence of this omission underestimates the natural frequency, ω_N, and damping, ξ, of the rotation axis. Referring to the linearized 3DOF in Appendix A, with the inclusion of the translation axis these become:

$$\omega_N = \sqrt{\frac{C_1 + C_2\, C_3}{I}} \tag{3}$$

$$\xi = \frac{C_2 + C_3 I}{2\sqrt{I\left(C_1 + C_2\, C_3\right)}} \tag{4}$$

Where: C_1 is the corrective moment constant, = Q*CNa*Sref*Zs, C_2 is the damping moment constant, = Q/V*DMC+mdot*Ln^2, and C_3 is the flight path pole, = Q*Sref*CNa/(mass*Vy). Note that if C_3 is set to zero the results equal the Mandell expressions. The main effect of the translation axis is to increase the natural frequency and, more importantly, the damping factor. Without consideration of C_3 one would conclude that a rocket with high moments of inertia will have damping factors near zero. Our results show this is not the case and heavy rockets are well damped by the translation axis. The 3DOF simulation confirmed this result.

The rotation and horizontal axes may be estimated from a small angle approximation of the full 3DOF equations and is presented in Appendix A. From Appendix B, the small angle approximation yields a very simple result for the horizontal acceleration (B1):

$$A_X = \frac{L + F\theta}{m} \tag{5}$$

Where L is the lift force, = Q*Sref*CNa, F is the thrust, θ is the body angle and m is the mass. Our problem will be to estimate the value to place on these parameters for each disturbance such that the average value of that acceleration during the engine burn is correct. The horizontal velocity may be estimated assuming a constant acceleration during the burn time, t_b, as:

$$V_X = A_X\, t_B \tag{6}$$

The flight path angle, γ, may be approximated as:

$$\gamma = \frac{V_X}{V_{YB}} \tag{7}$$

The vertical velocity at burn out, V_{YB} is given in equation (C2) in Appendix C.

Effects of Wind

This derivation assumes that the lift forces will average zero and the body angle will equal the peak angle of attack due to the wind. The results from Appendix B, equations (B3) and (B11) show the peak angle of attack and horizontal velocity to be:

$$a_{WIND} = \tan^{-1}\left(\frac{V_{WIND}}{V_{TIP}}\right) \quad (8)$$

$$V_{XWIND} = sign(Z_S)\frac{V_{WIND}\,K_{AW}\,I_T}{V_{EFFWIND}\,M_T} \quad (9)$$

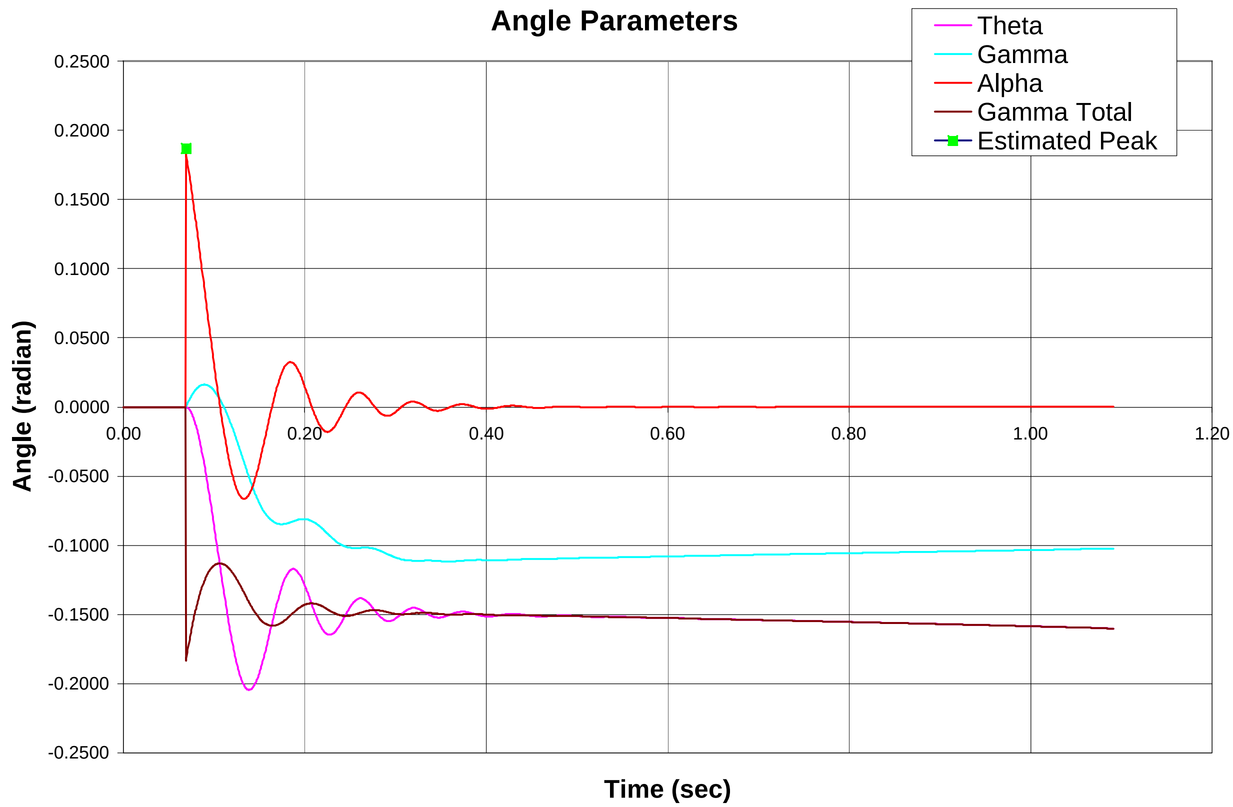
The effective velocity for the wind is given in Appendix B, equations (B7) and (B8). Note that the flight path angle, γ, can be estimated as $\gamma = V_{XWIND}/V_{YB}$. The effective velocity is the sum of the tip off velocity and an excess velocity that characterizes the dynamics of the wind. Both the peak angle of attack and horizontal velocity are inversely related to the tip off velocity, Vtip.

It is obvious that the length of the launcher will play an important part in the optimization of performance in wind. The higher the tip off velocity (a longer launcher), the greater the altitude. In our example, doubling the launcher length form .914 m to 1.83 meters reduced the peak angle of attack from .187 radians to .133 radians. Note that .0175 radian = 1 degree. The horizontal velocity was reduced to 10 m/s from 14 m/s.

Another parameter that effects wind performance is the excess velocity, given in equation (B7). This complicated expression contains parameters related to CNa, therefore, fin design. Careful examination of this function will show that the maximum altitude is achieved with the smallest possible fin size (near zero static margin). Unfortunately, small size fins reduce the point of positive pitching moment. Thus, a critical wind speed is reached where the rocket becomes unstable. Our top down procedure will ensure that the critical wind speed is properly handled.

In the special case of wind, it is readily apparent from Figure 1 that the cosine altitude penalty may be avoided by tilting the launcher away from the wind by an amount equal to γ. The rocket will turn into the wind and will fly straight up. This can improve altitude performance by as much as 5-7% under high wind conditions. The altitude loss due to turn energy cannot be avoided. The higher the tip off velocity, the greater the turn energy. Surprisingly, the turn energy is not a strong function of launcher length. The magnitude of the turn is smaller which compensates some of the loss. That is because the peak angle of attack and L/D are also related to launcher length and everything nearly cancels out the increase due to velocity.

The validation of the approximation for the effects of wind is shown in Figure 3 below. See Appendix B for the parameters used for the rocket design and 3DOF simulation. The 3DOF simulation and the estimates are shown for angles and horizontal parameters for a 5 m/s wind. Excellent agreement between the simulation and the estimates is indicated by the dot showing



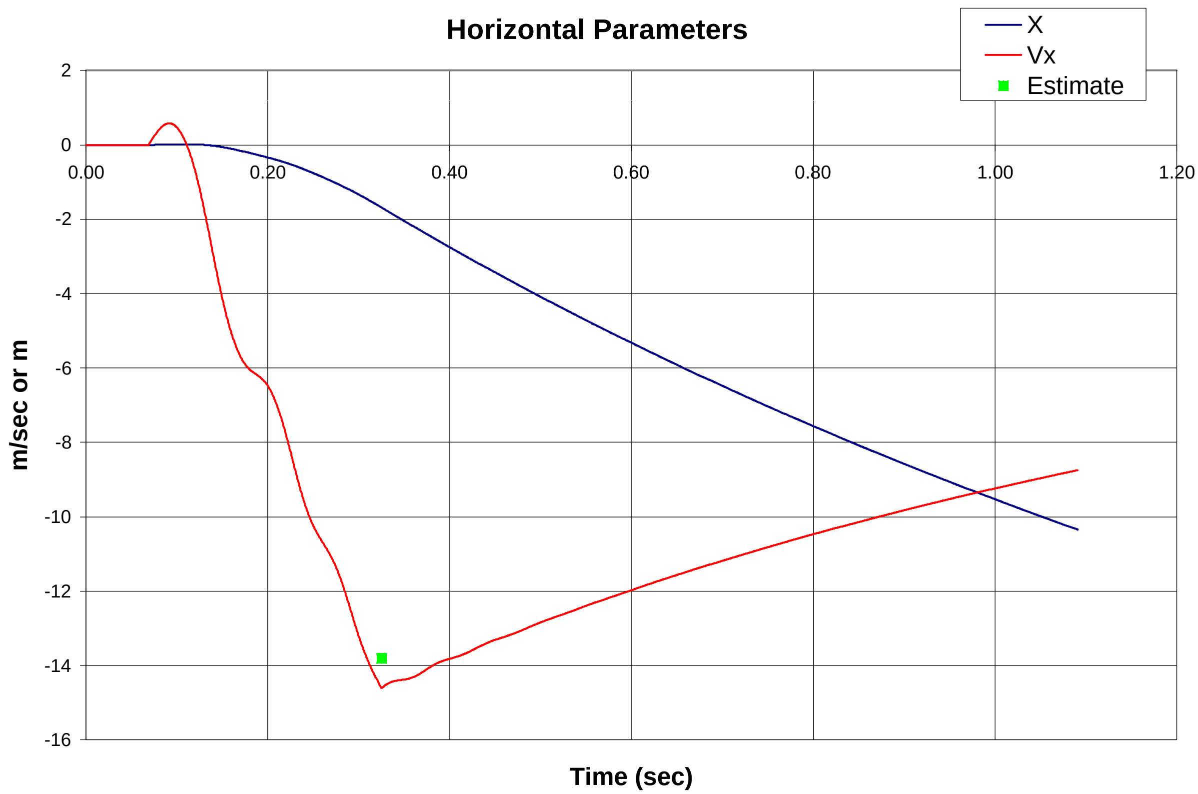


Figure 3 Wind Performance

the estimate at the peaks of the angle and velocity. Note that the angles settle out quickly in about .3 seconds. That is only .25 seconds from tip off. The velocity reaches a maximum at engine burn out which, in this case, is also about .3 seconds.

Effects of Airframe Errors

Airframe errors may be approximated as an induced angle of attack. For the typical model rocket, the largest source of error is probably due to fin misalignment. We have not attempted to model the details of fin errors, since the errors may be held to small numbers compared to the other disturbances. We assume that the angle of attack is approximately equal to the fin misalignment. That will allow us to estimate the accuracy required for the fins. A single fin that is misaligned will cause the rocket to roll. In order to sustain an angle of attack, two fins have to be misaligned the same way. We also assume that the body angle will become twice the peak angle of attack. We will again assume that the lift will average zero. From Appendix B, the peak angle of attack and horizontal velocity (B16) are:

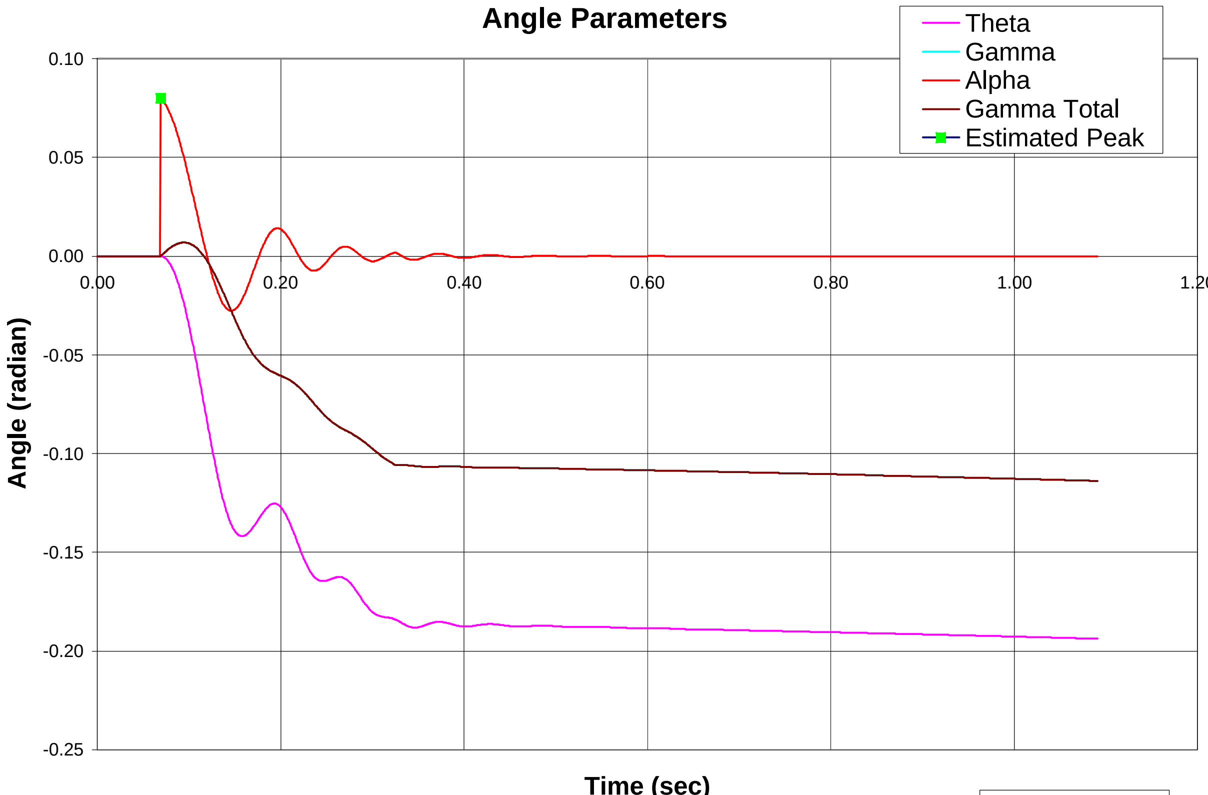
$$\alpha_{PEAK} = \psi \qquad (10)$$

$$V_{XAERO} = \frac{sign(Z_S)\, 2\psi \; I_T \; K_{AA}}{M_T} \left(\frac{t_B - \dfrac{1}{2F_N}}{t_B} \right) \qquad (11)$$

ψ is the airframe misalignment angle. When the burn time, tb, exceeds the wavelength, (1/Fn), by a substantial amount, the time correction term will be near one. The strategy to maximize altitude is simply to keep the airframe errors small. Small, in this case, means compared to wind and thrust errors. In most model rockets, that should not be difficult to achieve.

For validation with the simulation, we chose an airframe error large enough to approximately equal that of the 5 m/s wind. In this case, the error turns out to be about .08 radians (4.5 degrees). The usual airframe error might be less than one degree, demonstrating that airframe error might be small compared to other errors. The validation is shown in Figure 4 below. Again, good agreement with the 3DOF is demonstrated. Note that the angles are somewhat different than for the wind and that γ=γt since the wind is zero. The angles continue to build up as long as the engine is firing. That was not the case for wind. It is clear that the body angle is about twice the flight path angle, as predicted by theory.

The launcher length has a minimal effect on the airframe errors. In our case, increasing the launcher from .914 meters to 1.83 meters does not change the peak angle of attack and reduced the horizontal velocity from 14 m/s to 12 m/s.



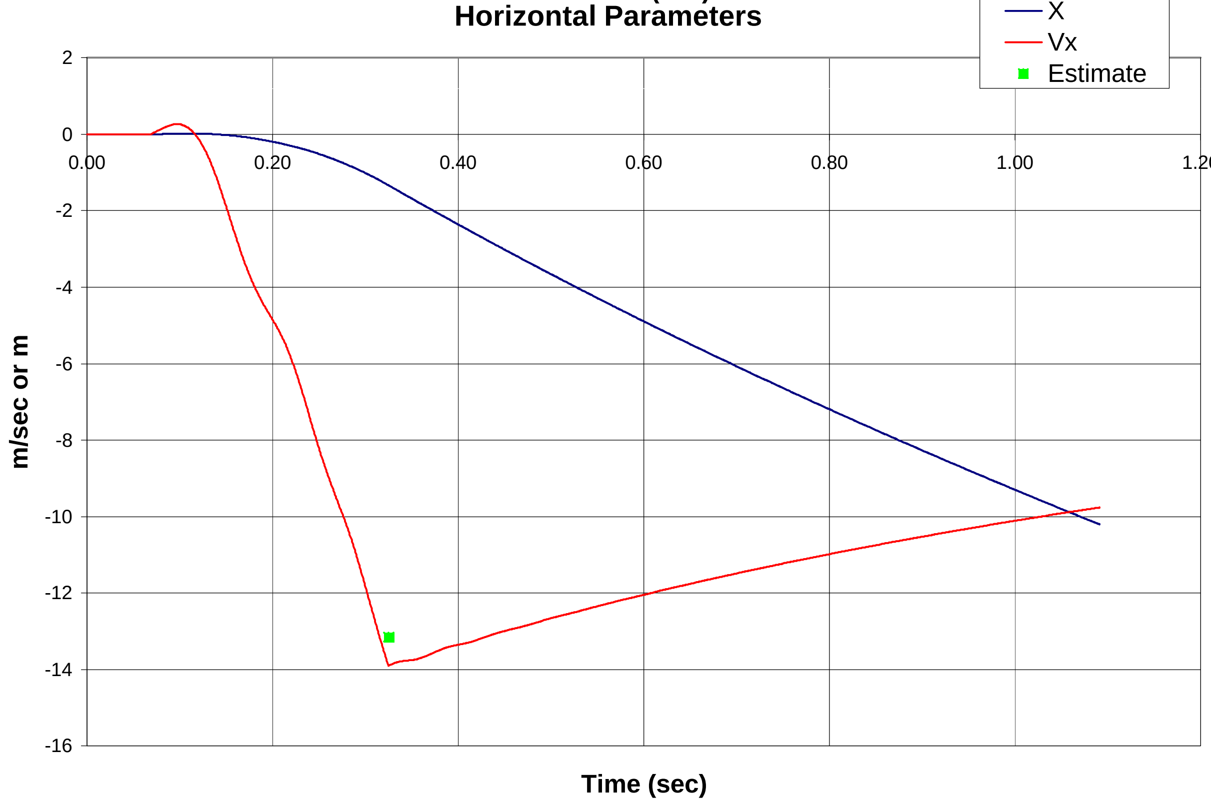


Figure 4 Airframe Performance

Effects of Thrust Misalignment

Thrust related disturbances are caused when the thrust vector does not go through the center of gravity. That will cause a pitching moment. Thrust misalignment may be caused by a number of sources including engine misalignment to the airframe, a failure of a cluster to fire all engines, as well as thrust errors in the engine itself. Because the engine can generate large forces, this disturbance can be quite significant, even for very small errors. Here we assume that the body angle will become twice the peak angle of attack and the lift forces will average out to produce a pitching moment equal to the thrust torque. From Appendix B (B20):

$$\alpha_{PEAK} = \frac{2\,F_B Tan(\beta)\,L_N}{Q\,CMC} \tag{12}$$

$$V_{XTHRUST} = \frac{-\,sign(Z_S)\,I_T\;Tan(\beta)\,L_N}{M_T}\left(\frac{2\,F_B}{Q\,CMC} - \frac{1}{Z_S}\right)\left(\frac{t_B - \dfrac{1}{2\,F_N}}{t_B}\right) \tag{13}$$

β is the thrust misalignment angle and L_N is the distance from the Cg to the end of the nozzle. Thrust errors are unlike wind and airframe errors because there is an optimum fin size that will maximize altitude. That is because both the peak angle of attack and the horizontal velocity are inversely related to CNa which is contained in CMC (CMC = CNa Sref Zs). CNa is related to the size and position of the fins.

For thrust misalignment, larger fins will yield a higher altitude, until the point is reached where the extra drag starts to limit performance. It is clear that there is a trade off to be made when considering the effects of thrust errors. For wind and airframe, the smallest fins generate the best performance. Small fins will give up performance to thrust error. Since we can fly with only one fin size, it is clear that a compromise will be made to achieve the best performance. The extent of the trade off will be dependent on the exact specifications for wind, airframe and trust errors.

For validation with the simulation we chose a thrust error large enough to approximately equal that of the 5 m/s wind. In this case it turns out to be about -.011 radians (0.625 degrees). The results are shown in Figure 5 below. Good agreement is demonstrated between the estimates and the 3DOF. The angle continues to build up as long as the engine is firing, as with airframe errors.

The optimum altitude with fin span is plotted in Figure 6 below. A comparison between the estimate and the 3DOF, shows that there is static altitude difference of about 1 meter. This is due to the small errors in the estimate. The estimated maximum altitude occurs for a fin span of 17 mm. The 3DOF shows the maximum at about 17.5 mm, 3% higher. That is not too bad, considering the complexity of the thrust errors. A 17 mm span on our special test vehicle represents a static margin of only .36 caliber. That may not be a practical design point for this small rocket because of variations in engine mass. An Estes 1/2A3 engine can vary in mass between 5.7 and 6.9 grams. That will cause a shift in Cg of about the same percentage and seriously degrade performance. This would require the modeler to sort engines or rebalance each flight. In addition, the Cp can move forward during transonic flight more than one caliber.

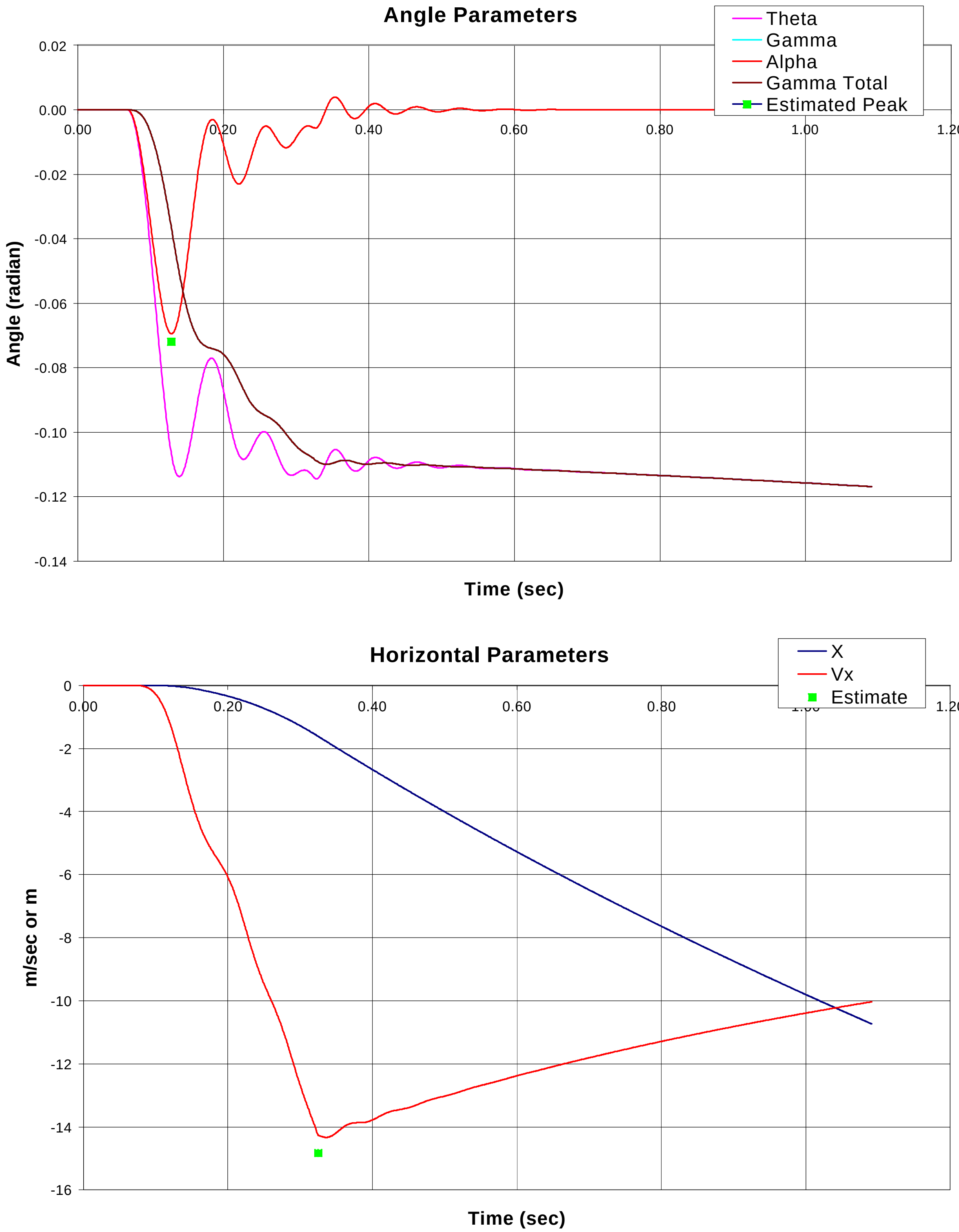


Figure 5 Thrust Performance

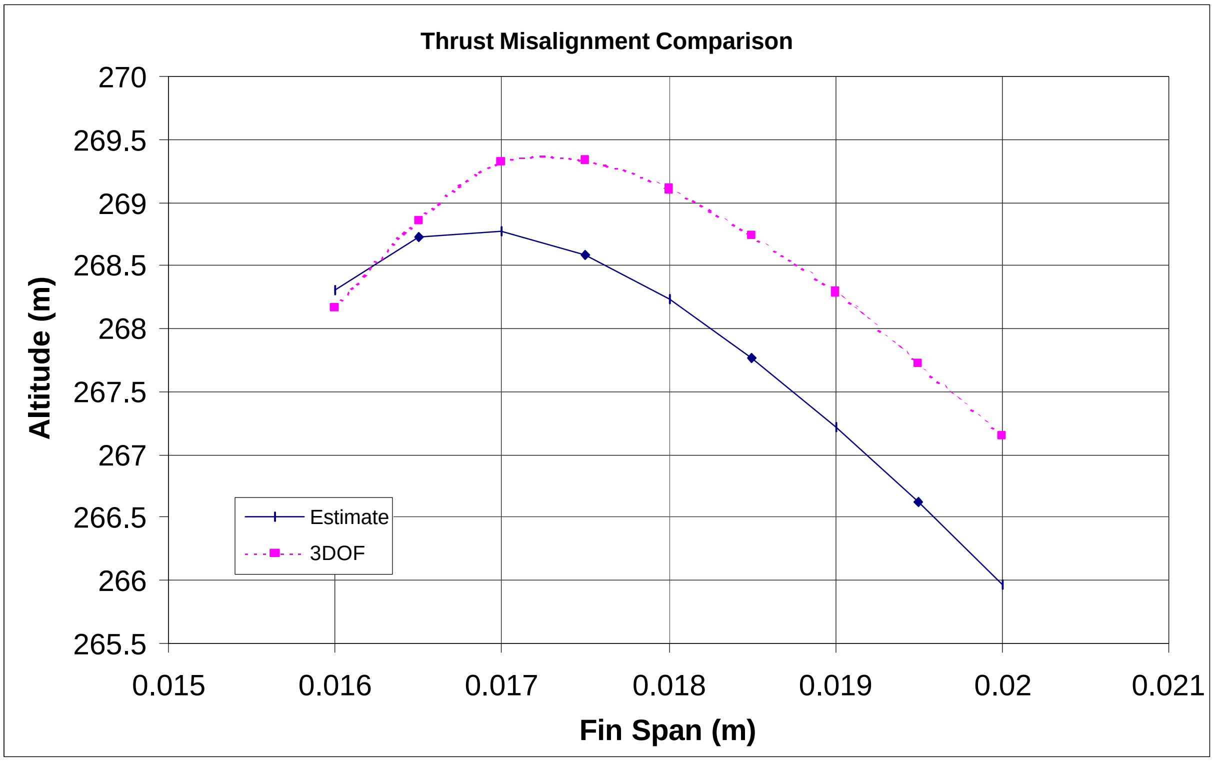


Figure 6 Thrust Altitude Verses Fin Span

Therefore, care should be taken to provide adequate margin for these and other factors that may degrade the assumptions used for design. In our example, a fin span of 20 mm will degrade the maximum altitude by 1% and will yield a static margin of about .6. The same rocket with a one-caliber margin would require a fin span of 30 mm and that would reduce the altitude to about 250 meters.

The launcher length moderately effects the thrust errors. A doubling of the launcher length causes little change to the peak angle of attack and reduces the horizontal velocity to about 11 m/s from 14 m/s.

Combination of Effects

When considering combinations of the various disturbances it important to know how to add the effects. For the case of horizontal velocity and γ, all of the effects add together directly. So in a worst case condition it will be necessary to total the individual effects.

The peak angle of attack is different. The angles for wind and airframe may be added as they both have the peak at the instant of tip off. This is because there is a step angle of attack at the end of the launcher. Thrust misalignment generates a step torque at the launch tip and the angle of attack must build up after tip off. The peak angle of attack due to thrust misalignment occurs at a time of about a quarter wavelength after tip off. As a consequence, the peak angle of attack is the maximum of the sum of wind and airframe or thrust misalignment.

Discussion of Results

We have developed a simplified set of estimates of the effects of wind, airframe and thrust misalignment. The overall validation to the 3DOF is quite good. Figure B1 in Appendix B confirms that the difference between the estimate and the 3DOF has a standard deviation of less than 1 meter (about 5%). Common to all of these effects is that the horizontal velocity is directly proportional to the total engine impulse. Since the vertical velocity is also proportional to total impulse one would not expect the flight path angle, γ, to be significantly affected by engine selection. However, the tip off velocity is very significant and is dependant on engine selection, therefore any significant effect on γ will be due primarily to tip off velocity caused by a particular engine selection.

The selection of the launcher length and the launch angle are important parameters affecting altitude performance. There may be large differences in peak engine thrust between various engines which will influence launch length selection. The optimum launcher length may be found by plotting the maximum altitude as a function of launcher length. Using the parameters in our examples above we have generated such a graph in Figure 7 below.

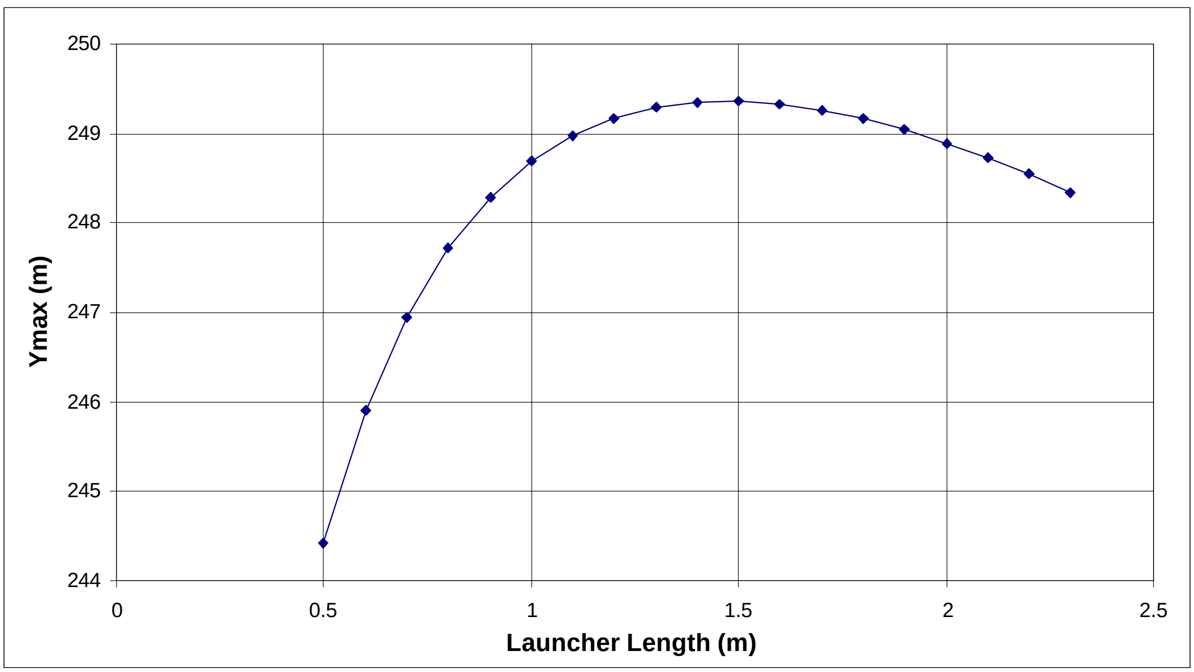


Figure 7 Effects of Launcher Length

Note that we have neglected launcher friction, as it is small compared to turn energy. For our model rocket, the optimum launcher length is about 1.5 meters. One can see how quickly the performance degrades with too short a launcher. The optimum launch angle is determined by measuring wind speed and tilting the launcher with the wind by an amount equal to the estimated flight path angle caused by wind.

Design Procedure

In order to use our design procedure, the modeler must have available the means to estimate basic rocket parameters such as the location of the Cg, Cp, air density with altitude and engine thrust curves. Many of these calculations may be done by hand or by using software such as Rocksim® or similar programs. Use the following steps to arrive at a baseline design. When designing for maximum altitude, keep the following rules in mind.

- Keep the overall length as short as possible, however, leave ample room for recovery devices and tracking powder.
- Fly with the optimum mass that will maximize altitude. Plot Ya for various values of mass to locate the correct value. This may not be possible for engines below B. Use tracking powder as the ballast to arrive at the correct mass. Keep the mass as far forward as possible.
- Keep the fin design as compact as possible. We like to use equal root and tip cords set to about one body diameter. Angle the fins aft by 20-40 degrees, we like 30. This will keep the Cp further aft.
- Keep drag to a minimum. Prepare the surface with a good slick finish. Use a tower launcher rather than lugs.
- Select low thrust/ long burning engines. Try to keep the rocket from going supersonic. This wastes a lot of energy due to drag divergence. Limit the speed to below Mach .9.
- Keep the fins as straight as possible. It is not likely that spinning the rocket will prevent the turn at tip off because of the low airspeed. Do not waste energy trying to make it spin. This is fruitless since most of the errors occur within one wavelength of the end of the launcher. The rocket will not have sufficient airspeed to start spinning before the turn. It will spin after it makes the turn and only waste energy. However, a small amount of roll can reduce subsequent altitude loss on long burning engines.

Procedure

1. Finish the complete rocket design, except for the fins. Estimate the worst case aft location for the Cg. This should consider engine weight variations, shifts in location due to parachute/ streamer location and any other parameters that will affect Cg.
2. Select specifications for the maximum wind, airframe and thrust misalignment. Good targets might be 10 m/s for wind and .018 radian (1 degree) for airframe errors. Estimate the worst case thrust misalignment if there is a static error due to the design. Additionally, consider effects of random variables such as engine attachment/alignment errors. For example, a 13mm engine will often have a clearance with the body tube of about .1 to .2 mm (.005 to .01 inch). One can assume the engine may not be centered by this amount. Allow for engine thrust misalignment due to errors in the nozzle and/or nozzle erosion. Our experience suggests that a maximum value for small black power engines might be as much as .013-.018 radians (0.75-1 degree) and composite engines may be less than 1/3 of those values.

3. Select a maximum limit for the angle of attack to be experienced during flight. Calculate this value using our method in reference [6] or use a conservative value of 15 degrees.
4. Find the optimum fin size. Calculate the drag of various candidate fin spans. Using the drag for each fin plot the maximum altitude for each fin span, assuming the values for above for wind, airframe and thrust error. Check the static margin at the worst case Cg location. Estimate the Cp location using the Barrowman method. If the rocket experiences transonic flight, increase the fin span to add an additional margin of about 5% of the body length.
5. Find the optimum launcher length by plotting Ymax for various launcher lengths. Then verify that the maximum angle of attack does not exceed the specification for the optimum launcher length. If it does, increase the launcher length or fin span until it does. Recheck the optimum fin span with the optimum launcher length. Readjust if necessary.

Conclusions

This R&D report has demonstrated a design procedure may be developed that will maximize altitude in the presence of wind, airframe and thrust errors. For a given set of specifications on wind, airframe and thrust errors, there is a fin design that will maximize altitude performance. Simple approximations to the actual performance provide easy means to estimate results and provide insight into the various effects. The analysis and design procedure results in the following important conclusions:

- There is an optimum fin size which will maximize altitude in the presence of combinations of thrust and wind errors.

- The launcher length and angle are important in wind and thrust errors. There is an optimum launcher length. The error due to wind may be minimized by tilting the launcher with the wind by an amount equal to the estimated flight path angle, γ, caused by the wind.

- The optimum fin size due to the effects of wind and airframe errors is as small as possible, but not allowing the peak angle of attack to exceed 15-20 degrees. Small fins also may have very small angles of attack to reach unstable pitching moments. Thus making fins small to avoid wind errors may make the rocket unstable at a low wind speed. The launcher length should be selected to keep the peak angle of attack below 15-20 degrees.

Equipment and Cost

There were no equipment costs associated with this R&D project.

Acknowledgments

We thank Jim Barrowman for all his help in understanding the effects of finless rockets and his willingness to suffer with fools like us. Special thanks to Jim for his critique of this paper. Thanks to Gordon Mandell for his review, comments and moral support for this work. Thanks to Robert LaBudde for his help sorting out the problems with the simulation. We also wish to thank Billie J. LaBudde and Edward Terry for all of their editorial help. And to Alan Jones for his helpful suggestions.

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Appendix A
3DOF Model

A three degree of freedom simulation (3DOF) will be developed in order to validate the approximations developed for the estimation of performance. The three degrees of freedom are the rotational axis (pitch), the vertical (Y) and horizontal (X) translational axes. The 3DOF allows an accurate prediction of the dynamic effects during rocket launch. The assumptions that went into this 3DOF include:

- zero roll rate
- α is replaced with $\sin(\alpha)$ in aerodynamic equations
- Air density is computed on a 1976 standard atmosphere
- Euler integration is used
- Wind, airframe and thrust errors are zero until tip off
- The rocket is launched straight up

A nonlinear normal force coefficient, CNa_alpha, based on our previous work [6] is used in place of the CNa.

3DOF equations

Drag Coefficient,	$Cd = Cdo \cos(\alpha)+CNa\_alpha \sin^2(\alpha)$
Lift Coefficient,	$Cl = -Cdo \sin(\alpha)+ CNa\_alpha \sin(\alpha) \cos(\alpha)$
Drag Force,	$D = Q\ Sref\ Cd$
Lift Force,	$L = Q\ Sref\ Cl$
Reference Area,	$Sref = \pi$ Nose Diameter$^2/4$
Horizontal Acceleration,	$Ax = (-D \sin(\gamma t) +L \cos(\gamma t)+F \sin(\theta))/m$
Vertical Acceleration,	$Ay = -g + (-D \cos(\gamma t)-L \sin(\gamma t)+F \cos(\theta))/m$
Horizontal Velocity,	$Vx = Vx +Ax\ dt$
Vertical Velocity,	$Vy = Vy +Ay\ dt$
Total Velocity,	$Vt = sqrt((Vx-Vw)^2 +Vy^2)$
Dynamic Pressure,	$Q = .5\ \rho\ Vt^2$
Horizontal Position,	$X = X +Vx\ dt$
Vertical Position,	$Y = Y +Vy\ dt$

Body Angular Acceleration,	$\theta'' = (F\ miss-Q\ CMC \sin(\alpha) - (Q/Vt\ DMC +mdot\ Ln^2)*\theta')/I$
Body Angular Velocity,	$\theta' = \theta' + \theta''\ dt$
Body Angle,	$\theta = \theta + \theta'\ dt$
Corrective Moment Coeff,	$CMC = CNa\_alpha\ Sref\ Zs$
Static Margin,	$Zs = Cg-Cp$
Damping Moment Coeff,	$DMC = Sref\ (Cnn\ (Cg-Xn)^2+Cnt\ (Cg-Xt)^2+Cnf\ (Cg-Xf)^2)$
Thrust Misalignment,	$miss = ln*\tan(\beta)$
Angle of Attack,	$\alpha = \theta - \gamma t$
Flight Path Angle in Wind,	$\gamma t = atan2(Vy, Vx-Vw)$
Flight Path Angle,	$\gamma = atan2(Vy, Vx)$

Dynamic Normal Force Coefficient

This is taken from our paper "Extending the Barrowman Method to Large Angles of Attack," using the model in AERO_97.XLS

Moment Normalization Factor, MNF = -Abs(Cg - Cpo)
Maximum Value of Cm at Stall, Cm_peak = Sign_α * (Cg - (Cpo + Sin(Abs(α_stall)) * (CLA - Cpob))) * Cna * Sin(α_stall) / MNF
Maximum Value of Cmb at Stall, Cmb_peak = Sign_α * (Cnaob + Sin(Abs(α_stall)) * (Cnac - Cnaob)) * (Cg - (Cpob + Sin(Abs(α_stall)) * (CLA - Cpob))) / MNF * Sin(α_stall)
Corrected Cp With Fins Attached, Cpf = Cpo + Sin(Abs(Alpha)) * (CLA - Cpob)
Corrected Cna of Body Alone, Cnab = Cnaob + Sin(Abs(Alpha)) * (Cnac - Cnaob)
Corrected Cp for the Body Alone, Cpb = Cpob + Sin(Abs(Alpha)) * (CLA - Cpob)
Picthing moment of Body Alone, Cmb = Cnab * (Cg - Cpb) * Sin(Alpha) / MNF

If Abs(Alpha) <= α_stall Then
Cn = Cna * (Cg - Cpf) / MNF
Else
Cn = (Cmb + Cm_peak - Cmb_peak) / Sin(Alpha)
End If

Cna_alpha = Cn

Where:

Parameter	**Symbol**
Thrust Misalignment Angle	β
Drag Coefficient At Zero α	Cdo
Center Of Gravity	Cg
Center Of Pressure Of Body Alone By Cutout Method	CLA
Normal Force Coeff of Total Rocket	Can
Normal Force Coeff For Body Alone	Cnaob
Normal Force Coeff For Body Alone By Cutout Method	Cnac
Normal Force Coeff For Nose Transition And Fin	Cnn, Cnt, Cnf
Center Of Pressure of Total Rocket	Cpo
Center Of Pressure Of Body Alone	Cpob
Time Step	dt
Thrust	F
Gravity	g
Angular Moment Of Inertia	I
Rocket Mass At Time t	M
Mass Flow Rate	Mdot
Air Density	ρ
Center Of Pressure For Nose Transition And Fin	Xn, Xt, Xf

Linearizing the 3DOF equations into a 2DOF

It will be useful to reduce the complexity of the 3DOF to a simplified two degrees of freedom (2DOF) so that approximations to the dynamic performance may be undertaken. The 2DOF will include the rotational axis and the horizontal translation axis (X). We recast the 2DOF equations by neglecting the vertical axis (all we need is the velocity Vy), assuming the rotational axis is the same as 3DOFand by simplifying the X axis with small angle approximations. We will also ignore the horizontal position. This leads to:

Lift Coefficient,	Cl = CNa α
Lift Force,	L = Q Sref Cl
Horizontal Acceleration,	Ax = (L +F θ)/m
Horizontal Velocity,	Vx = Vx +Ax dt

It is useful to visualize the 2DOF equations in block diagram form. These are shown in the

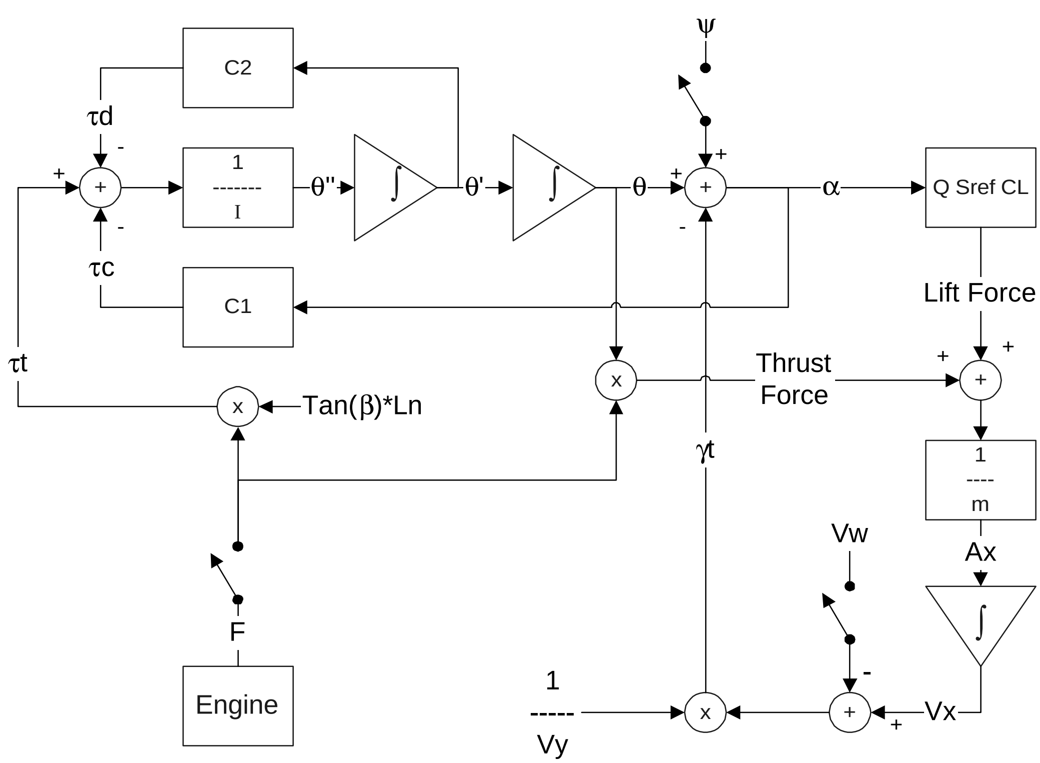

Figure A1 Block Diagram of 2DOF

figure above.

The circles with the “X” in it is a multiplier, with a “+” means adder with the proper sign shown. The switches represent the condition at tip off where the parameters are “turned on.”

Note that the angle of attack is subject to two feedback loops. The first is the rotation axis and the second from the translation axis, which generate γt. Both θ and γt produce α.

In the diagram the constants for C1 and C2 are the Mandell notation for:

C1 = Q*CMC
C2 = Q/V*DMC+mdot*Ln^2

We may note that the loop gain around the translational loop may be denoted as C3. The value for C3 is found by taking the products of all the elements around the loop:

C3 = Q*Sref*CNa/(m*Vy)

C3 is also known as the "flight path pole." This is the lag between the flight path angle and the angle of attack.

The natural frequency and damping factor of the combined loops is found to be:

Wn = sqrt((C1+C2*C3)/I)

ξ = (C2+C3*I)/(2*sqrt(I*(C1+C2*C3)))

Note that if the translation loop is neglected (C3=0) the results reduce to the Mandell formulas.

Appendix B
Simplified Estimates of Dynamic Performance

We may estimate the effects of wind, thrust misalignment and airframe errors on rocket performance by using a linearized model of the equations of motion. We will use the 2DOF in Appendix A for the approximation of performance. The linearized x axis acceleration becomes:

$$Ax = (L+F\ \theta)/m \qquad (1)$$

We also recognize that maximum x velocity will be at engine burn time, tb, and we assume that the acceleration is constant, representing the average value during thrust, thus the velocity becomes:

$$Vx = Ax\ tb \qquad (2)$$

We will assume that the disturbance begins at the end of the launch rail. Further simplification results by neglecting the oscillations present during tip off by replacing the dynamic parameters with the average value. This assumption is valid because the time scales usually exceed one cycle of oscillation and because the motion involves several integrals which will average out the oscillations. Our challenge will be to select appropriate values for the parameters in the acceleration equations for each effect.

Effects of Wind

When the rocket reaches the end of the launcher, it experiences a step angle of attack due to wind. The rocket will weathercock into the wind, if it is stable. The peak angle of attack is defined as:

$$\alpha wind = ATAN(Vw/Vtip) \qquad (3)$$

where Vw is the wind speed and Vtip is the tip off velocity.

The maximum allowable wind speed for stable flight may be derived by rearranging (3).

$$Vwmax = Tan(\alpha max)*Vtip \qquad (4)$$

where αmax is the maximum allowable angle of attack.

The tip off velocity, Vtip, as given in Appendix C:

$$Vtip \approx SQRT(2*Yl*(Ft/Mt-g)) \qquad (5)$$

where Yl is the launcher length, Ft is the average thrust on the launcher, Mt is the total lift off mass, and g is the acceleration due to gravity.

Because the rocket is accelerating, the effective velocity that best characterizes the dynamic performance will be higher than the tip off velocity. An additional excess velocity, Ve, which must be added to the tip off velocity to produce the approximate result, will be required. We will assume that this excess velocity is a function of the "wavelength" of the rocket. Here wavelength refers to the reciprocal of the resonate frequency, $\lambda = 1/Fn$, of the rocket, and has the units of time. For most model rockets the damping will be small; therefore, we can neglect the influence of damping on resonate frequency. Fn may be estimated, neglecting the damping and the horizontal axis effects, as:

$$Fn \approx SQRT(Q*CMC/I)/(2*\pi) \quad (6)$$

Where Q is dynamic pressure, $Q=.5*\rho*V^2$,
CMC is the corrective moment coefficient, CMC=Cnao*Sref*Zs,
I is the longitudinal moment of inertia

We will define Ve as some fraction, Ke, of the wavelength by calculating the acceleration at lift off times the effective time scale (wavelength) of the rocket:

$$Ve \approx Thrust/(mass*Ke*Fn) \quad (7)$$

Since the resonate frequency is also a function of velocity, we must make the substitution and solve for Ve:

$$Ve\_wind = (SQRT(SQRT(CMC*\rho)*Ke\_w*Mt*Vtip^2+8*SQRT(2*I)*\pi*Fe)-SQRT(Ke\_w*Mt)*Vtip*CMC^{1/4}*\rho^{1/4})/(2*SQRT(Ke\_w*Mt)*CMC^{1/4}*\rho^{1/4}) \quad (8)$$

Therefore the effective velocity, Veff_wind, which best characterizes the dynamic conditions near tip off becomes:

$$Veff\_wind = Vtip + Ve\_wind \quad (9)$$

In the case of wind input, the constant, Ke_w, which best defines the wavelength is 5.

We will use this concept of effective velocity for the other sources of disturbances as well.

We are now ready to estimate the acceleration and velocity due to wind disturbance. We must return to equation (1). It will be necessary to estimate the parameters for lift, L, and body angle, θ. For a wind input the lift forces will be a damped sine wave centered about zero. The average lift force due to the wind transient will be approximately zero. We will assume that the body angle, θ, will take on the value of $\alpha \approx Vw/Veff$. These assumptions produce a horizontal acceleration of:

$$Ax\_wind \approx sign(zs)*Vw/(Veff\_wind*Mt)*Fb*Ka\_w \quad (10)$$

Where the sign(zs) indicates the turn into the wind if the rocket is stable and Fb is the average thrust over the burn time.

Note that we have added a scaling factor, Ka_w, which will allow us to improve our estimate to take into account any small errors in the assumptions. For the wind disturbance the 3DOF tells us that Ka_w ≈ .9 yields excellent results.

The peak horizontal velocity will occur at the end of the engine burn time, tb. Substituting into (2) yields the peak horizontal velocity, Vx_wind:

$$Vx\_wind = sign(zs)*Vw/(Veff\_wind*Mt)*Fb*Tb*Ka\_w \quad (11)$$

Recognizing Fb*Tb as total impulse, It (11) may be written as:

$$Vx\_wind = sign(zs)*Vw/(Veff\_wind*Mt)*Ka\_w*It \quad (12)$$

The flight path angle, caused by the wind, may be estimated from $\gamma \approx Vx/Vyb$, were Vyb is the vertical velocity at burn out derived in appendix C. Thus the flight path angle estimate for wind is:

$$\gamma wind = Vx\_wind/Vyb \quad (13)$$

Effects of Airframe Errors

Airframe errors are modeled as an angle of attack, ψ, and are produced by things such as fin misalignment. In many ways these errors are similar to wind effects, as they are seen as a step angle of attack at the launcher tip. We will use the same concepts used to develop the wind estimates. The excess velocity is the same as the wind (7) and therefore the effective velocity will be the same as the wind (9). Here $\alpha = \psi$. Then $\theta=\psi+\gamma$, and γ will become approximately ψ, the body angle will become about 2 ψ at the end of the burn time. Returning to (1) we again assume that the lift force will average zero, and the body angle will take on the value of 2 ψ we find that the horizontal acceleration becomes:

$$Ax\_aero=sign(zs)*2*\psi/Mt*Fb*Ka\_a \quad (14)$$

The above result assumes that wavelength will be much longer than the burn time. When that assumption is not true, full acceleration will not be achieved. This is because the dynamic response has not settled on the steady state value. A correction term must be added to account for this effect. The portion of the acceleration that will be achieved may be estimated by assuming that when the burn time is less than ½ the wavelength, the acceleration will be near zero. As a consequence there is not enough time for the system to settle. This results in a correction factor of (tb-1/(2*Fn))/tb, resulting in the final estimate of acceleration as:

$$Ax\_aero=sign(zs)*2*\psi/Mt*Fb*(tb-1/(2*Fn))/tb*Ka\_a \quad (15)$$

Note that in this case the scale factor, Ka_a, which produces the best fit to 3DOF data, is about .91.
The peak velocity due to airframe errors and the flight path angle are then:

$$Vx\_aero=sign(zs)*2*\psi/Mt*It*(tb-1/(2*Fn))/tb*Ka\_a \quad (16)$$

$$\gamma aero = Vx\_aero/Vyb \quad (17)$$

Effects of Thrust Misalignment

The effects of thrust misalignment are somewhat different than the effects of wind and airframe errors. Thrust misalignment causes a torque on the airframe as long as the engine is burning. This means that the lift forces will not average to zero as before. Also, the peak angle of attack occurs somewhat after tip off. The assumptions necessary for equation (1) are: the lift forces are driven by the thrust misalignment torque, $\tau t = Fb*Tan(\beta)*Ln$, where β is the misalignment angle and Ln is the distance from the Cg to the end of the nozzle. The lift force then becomes: $L=Q\ Sref\ CNa\ \alpha$. The steady state angle of attack due to thrust torque is then, $\alpha=(Fb*Tan(\beta)*Ln)/(Q\ CMC)$. We will assume that the peak body angle, θ, will take on the value of $2\ \alpha$, due to the low damping factor. Because the angle of attack requires time to build up, the same argument about the relationship of burn time to wavelength may be made as in the airframe derivation. Additionally, the wavelength scale factor, Ke_t, will be about 2 rather than 5 as before. Substituting these assumptions into equation (1) leads to:

$$Ax\_thrust = -sign(zs)*(Q*Sref*can*\alpha+Fb*2*\alpha)*(tb-1/(2*Fn))/tb*Ka\_t/Mt \quad (18)$$

Substituting the value for α and reducing yields:

$$Ax\_thrust = -sign(zs)*Fb*Tan(\beta)*Ln*(2*F/(Q\ CMC)- 1/zs) *(tb-1/(2*Fn))/tb /Mt \quad (19)$$

The peak velocity from (2) and flight path angle then become:

$$Vx\_thrust = -sign(zs)*It*Tan(\beta)*Ln*(2*F/(Q\ CMC)-1/zs) *(tb-1/(2*Fn))/tb /Mt \quad (20)$$

$$\gamma thrust = Vx\_thrust/Vyb \quad (21)$$

Note that comparison to 3DOF indicates that Ke_t should be about 1.5 and Ka_t about 1.29 for best accuracy.

Addition of Effects

The peak velocity for combinations of effects may be simply added together. Peak angles of attack cannot be added together, because they occur at different times. The angle of attack from wind may be added to the airframe since it occurs at tip off. The peak angle of attack will be the greater of the sum of wind and airframe of thrust.

3DOF Validation

A special rocket configuration was devised to test validity of the estimates. It is a small rocket which has accrued a great deal of actual flight test data. The design parameters for the test case are given in Table 1 below.

Table 1 Validation Rocket Design Parameters

Parameter	Value	Unit
Length Of Nose	0.031	m
Diameter Of Nose	0.013818	m
Total Length Of Rocket	0.111	m
Distance To Front Of Transition	0.031	m
Diameter At Rear Of Transition	0.013818	m
Length Of Transition	0	m
Distance To Leading Edge Of Root Cord	0.094	m
Root Cord	0.012	m
Tip Cord	0.012	m
Fin Span	0.02	m
Distance To Fin Tip	0.01154	m
Number Of Fins	3	NA
Center Of Gravity	0.0832	m
Nose Shape	Parabolic	Na
Stall Angle Of Attack	11	Degree
Drag Coefficient For The Body (cutout)	0.42	NA
Airframe Mass	0.00326	kg
Longitudinal Moment of Inertia	1.300E-05	kg-m^2
Distance to Nozzle	0.03	m
Drag Coefficient	0.38	NA

Note that there are two dependencies built into the parameters. First, the distance to fin tip is a 30 degree angle or Xs=.577 Span. Second, the Drag coefficient is related to fin span as Cd=.29+4.5*Span. In order to simplify the calculations, the engine thrust is assumed to be constant throughout the burn time with a nominal value for a ½ A3 engine.

The 3DOF was used to validate the estimates derived above. A simulation for each effect was used to generate a table of values. A comparison of the linearity errors was done by regression analysis in order to estimate the value for the scale factor constants in each estimate. Using these values allows for minimum error. In all cases, the scale factors were near 1, indicating our assumptions were close to correct. The result of this analysis is shown in Table 2 of values for the constants in the estimates.

Table 2 Constants Used For Validation

Constant	Value
Ke_w	5
Ke_a	5
Ke_t	1.5
Ka_w	.896
Ka_a	.909
Ka_t	1.288

By using the above table of constants, the robustness of the estimates was undertaken. This was accomplished by using a Monte Carlo simulation with a number of design parameters changing simultaneously. One hundred test runs where done while varying the following parameters over the range indicated in Table 3 with a uniform random distribution.

Table 3 Monte Carlo Parameter Ranges

Parameter	Low Value	High Value	Unit
Center of Gravity	.08	.086	M
Burn Time	.1625	.65	S
Average Thrust	1.9	7.6	N
Moment of Inertia	$4.1x10^{-6}$	$4.1x10^{-5}$	kg-m^2
Fin Span	.015	.05	M
Cross Wind	0	-10	m/s
Airframe Error	0	-.05	Radian
Thrust Error	0	.01	Radian

The results are shown in Figure B1 below.

The regression line indicates a near perfect relationship between the 3DOF estimate and the estimated parameters. This is because we used the scale factor constants to minimize the error, otherwise there would be about a 10% scatter on the data. The standard deviation of the error is about .5 meters. Note the scatter is larger at high velocities and is probably due to the breakdown of the small angle assumptions.

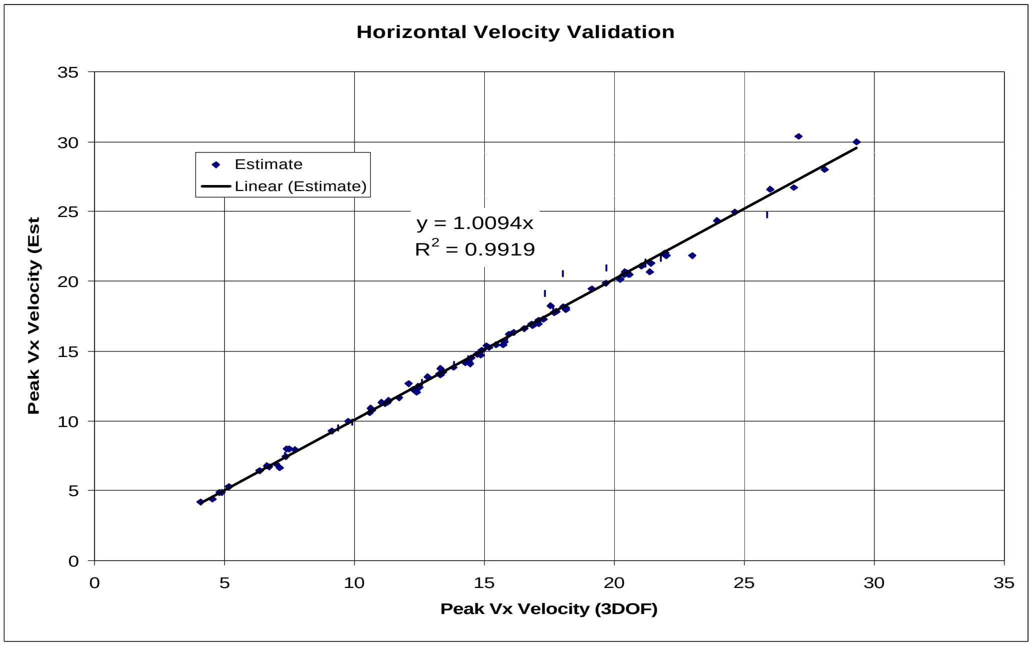

Figure B1 Monte Carlo Validation

Appendix C
Derivation of Vertical Velocity and Altitude Performance

We wish to approximate the vertical velocity and altitude performance during both the boost and coast phases in order to accompany our 2DOF estimates of the X axis performance.

Launcher Tip Off Velocity

While the rocket is on the launcher we can neglect the aerodynamic drag force because of the low speed. We can assume the total lift off mass, Mt, since very little of the fuel will be used. Further, we assume that the average thrust on the launcher, Ft, may be estimated from the engine thrust data and that the friction forces are small and can be neglected. This yields the tip off velocity, Vtip, given the launcher length, Yl as:

Vtip ≈ SQRT(2*Yl*(Ft/Mt-g)) (1)

Burn Out Velocity

The burn out velocity, in the presence of an aerodynamic drag force, may be obtained by solving the differential equations of motion using the method of separation of variables. Here we must assume that the thrust and mass are constant. We can use the average values to achieve these constraints. The general form of this method is

$$\frac{dv}{dt} = p(t)q(v)$$

where:

$$p(t) = 1$$

$$q(v) = -g + \frac{F - K_d V^2}{m}$$

Upon Integration

$$\int_{Vo}^{V} \frac{1}{p(v)} dv = \int_{to}^{t} p(t)\, dt$$

For the boost phase, we can assume that the initial velocity and time are 0, Fb is the average thrust, and mb is the average mass during boost. This yields the velocity at the end of the burn time, tb, of:

Vb = SQRT(Fb-g*mb)*(EXP(2*SQRT(Kd)*tb*SQRT(Fb-g*mb)/mb)- 1) /
(SQRT(Kd)*(EXP(2*SQRT(Kd)*tb*SQRT(Fb-g*mb)/mb)+1)) (2)

Where mb = Mt-1/2*mfuel

Kd = ½*ρb*Cd*Sref, where ρb is the average air density in the boost phase.

We are not the first to derive this result. Astute observers will recognize the exponential function as the hyperbolic tangent. Casting (2) in that form yields what has come to be known as the "Fehskens-Malewicki" solution. We do not have references for the original work, however, Viggiano [7] has a good discussion on this approach.

Burn out Altitude

The altitude at burn out may be obtained by integrating (2) with respect to time yielding:

$$Yb = mb*LN(((EXP(SQRT(Kd)*tb*SQRT(Fb-g*mb)/mb))^2+1)/(2*EXP(SQRT(Kd)*tb*SQRT(Fb-g*mb)/mb)))/Kd \quad (3)$$

Coast Altitude

Given the coast phase mass, mc, the altitude reached becomes:

$$Yc = mc/(2*Kd)*LN(1+Kd*Vb^2/(g*mc)) \quad (4)$$

Of course, the above value for Kd includes the average value of air density during the coast phase.

The coast time is then:

$$Tc = SQRT(mb/(g*Kd))*ATAN(Vb*SQRT(Kd/(g*mb))) \quad (5)$$

Total altitude at apogee is then Ya = Yb+Yc (6)

Total time to apogee is then Ta = Tb+Tc (7)

Altitude in the Presence of a Turn

The above results assume that the rocket is flying perfectly straight up. If the rocket is subjected to disturbance near the end of the launch tip, which results in the flight path angle γ being deflected during the turn, then the altitude will not be as high. There are two predominate reasons for the loss of altitude. First, and most foremost, the altitude is dependent on $\cos(\gamma)$; second, the turn will take energy away from the rocket, due to the extra drag generated by the turn. Thus the actual altitude achieved, Ymax, in the presence of a energy loss altitude becomes:

$$Ymax = Ya*\cos(\gamma) - Y_L \quad (8)$$

The altitude lost, Y_L, to turn energy may be estimated from the lift-to-drag ratio (L/D), the peak angle of attack during the turn, and the kinetic energy during the turn. Thus the energy consumed by the turn is:

$E_T = ½*Mt*Vtip^2*2*\alpha peak/(L/D) = Mt*Vtip^2*\alpha peak/(L/D)$ (9)

The L/D ratio may be approximated by assuming small angles as:
$L/D = (CNa*\alpha peak)/(Cdo+ CNa*\alpha peak^2)$ (10)

Please note that this approximation for energy loss will not give the correct result for angles of attack near zero. This is because as αpeak approaches zero, the L/D approximation will cancel the α term and will show constant energy loss even at zero αpeak. The error will only be a few meters in most cases.

The altitude loss due to the energy of the turn, using the coast mass, mc, then becomes just:

$Y_L = E_T/(mc*g)$ (11)

The values for αpeak and γ used in these equations for various sources of disturbances are estimated in Appendix B.