Triangulation for Altitude

A Practical Guide

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1 Introduction

The problem altitude determination based on the elevation and azimuth angles observed by two or more people on the ground has been discussed at some length by Stine[1] and others. What we'll do here is simplify and correct his discussion somewhat, then extend it in some detail to the cases he treats only briefly.

1.1 Geometry of Observation

In the first illustration, Fig. 1, the target (T) is observed along two lines of sight: S_1 and S_2 from positions O_1 and O_2. The point I, directly below the target, together with O_1 and O_2 forms a plane triangle on the ground. The line segment IJ, is constructed from the ground plane image (I) to the point J on the line D_{12} between the observers so as to meet this baseline at a right angle. The other sides of the ground plane triangle, D_1 and D_2, intercept the line IT at right angles as well. Thus, the altitude we seek, H_{target} is just a segment common to the two right triangles [T I O_1] and [T I O_2].

Given the ground plane triangle, elevation angles E_1 and E_2 are measured from that plane up to the lines of sight, S_1 and S_2. Defined in the ground plane, the azimuth angles A_1 and A_2 are measured from the baseline, D_{12}, to the segments O_1I and O_2I, respectively. Note that "D-one-two" means the distance between points one and two.

For an altitude measurement we need only the dataset: [D_{12}, E_1,A_1,E_2,A_2]; nothing more is required, and nothing less is acceptable.

In the second illustration, Fig. 2, the geometry encountered with three observers is shown. Here the view is down onto the ground plane, so that the image point (I) is all that is shown. The distances and angles as labeled in this case would allow the same formulae used for two observers to be extended to three, taken two at a time. The elevation angles for each observer would also be used in that same pairwise manner. There are, in other words, three possible datasets: [D_{12},E_1,A_1,E_2,A_2], [D_{13}, E_1,B_1,E_3,B_3], and [D_{23},E_2,C_2, E_3,C_3].

Fig.1 Basic Geometry

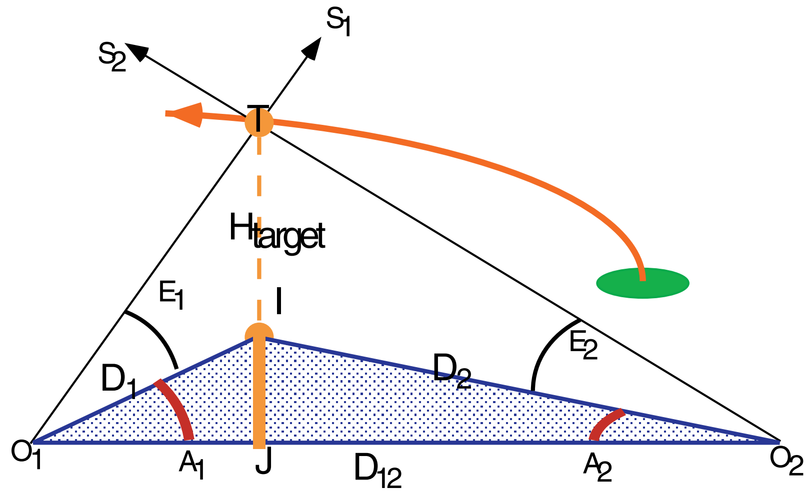
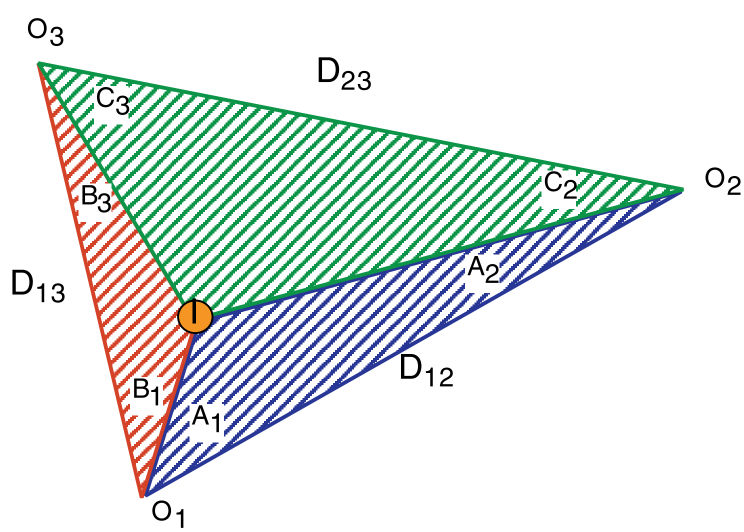


Fig.2 Multiple Observer Geometry



2 Development of a Two Observer Altitude Rule

What do we need from plane trigonometry to analyze these figures? First we can expect to be using the sin, cos, and tan functions as evaluated for various of the observed angles. Moreover, we need to evaluate these functions for particular sums of angles. Finally we need to organize the method into a stepwise plan or algorithm for the easy evaluation of the altitude formula from the data. In a field situation, where time is of the essence, one can expect the altitude measurement and data reduction to be ready within half a minute to a minute of a rocket's ascent.

2.1 Proof of the Altitude Formula

First, skip this section if you are not a math fan, all we need from this development is the final formula. Otherwise, you have entered the "theory zone". To begin, since we know (or at least hope) that the two lines of sight are converged on the same target, there are two equivalent formulations for the altitude:

$$H_{target} = D_1 \; tan(E_1) = D_2 \; tan(E_2) \; . \tag{1}$$

Next, since the point J in our baseline defines two right triangles, the length of the baseline is just the sum

$$D_{12} = D_1 \; cos(A_1) + D_2 \; cos(A_2) \; . \tag{2}$$

Similarly, just as for the common segment H, there are two formulations for the length of segment IJ: $\ell_{IJ} = D_1 \; sin(A_1) = D_2 \; sin(A_2)$, and so

$$0 = D_1 \; sin(A_1) - D_2 \; sin(A_2) \; . \tag{3}$$

Next, solve equations (2) and (3) for D_1 and D_2 in terms of the angles. Multiply (2) by $sin(A_1)$, multiply (3) by $cos(A_1)$, and then add to eliminate the term containing D_1, and you get

$$D_{12} \; sin(A_1) = D_2 \; [cos(A_2) \; sin(A_1) + sin(A_2) \; cos(A_1)] \; . \tag{4}$$

Multiply (2) by $sin(A_2)$, multiply (3) by $-cos(A_2)$, and then add to eliminate the term containing D_2, and you get

$$D_{12} \; sin(A_2) = D_1 \; [cos(A_1) \; sin(A_2) + sin(A_1) \; cos(A_2)] \; . \tag{5}$$

Now, the common term $[cos(A_1)\ sin(A_2) + sin(A_1)\ cos(A_2)]$ is just the $sin(A_1 + A_2)$. This gives expressions for D_1 and D_2

$$D_1 = D_{12}\ \left(\frac{sin(A_2)}{sin(A_1 + A_2)}\right)\ ;\ D_2 = D_{12}\ \left(\frac{sin(A_1)}{sin(A_1 + A_2)}\right)\ . \tag{6}$$

Substitute these expressions back into (1) and we get two equivalent calculations for the altitude

$$H_1 = D_{12}\ \left(\frac{tan(E_1)\ sin(A_2)}{sin(A_1 + A_2)}\right)\ ;\ H_2 = D_{12}\ \left(\frac{tan(E_2)\ sin(A_1)}{sin(A_1 + A_2)}\right)\ . \tag{7}$$

In practice these two versions of H will not be identical, due to pointing errors in the lines of sight and measurement errors in the angles. Both these estimates must remain positive, however, because **only one** of the interior angles A_1 or A_2 can exceed 90 degrees, and neither can their sum exceed 180 degrees. The usual practice is to check and see if H_1 and H_2 are within a given tolerance, say 0.1, and then average them.

So, in terms of something easy to check, **if** it is true that

$$\left(\frac{2|H_1 - H_2|}{[H_1 + H_2]}\right) \leq E = 0.1\ ,$$

then estimate the altitude as:

$$H_{target} = 0.5\ D_{12}\ \left\{\left(\frac{tan(E_1)\ sin(A_2)}{sin(A_1 + A_2)}\right)\ +\ \left(\frac{tan(E_2)\ sin(A_1)}{sin(A_1 + A_2)}\right)\right\}\ . \tag{8}$$

If the test fails, then the measurement is usually thrown out. It is certainly possible to be more precise about how much error is to be allowed, or whether to weight the two estimates differently depending on the steepness of the elevation angles. In model rocketry such hairs usually remain unsplit, but if the requirement for higher precision arises, then the addition of a third or fourth observer can drive the errors to very small values indeed.

2.2 Applications of the Two Observer Altitude Rule

Using our notation from the first illustration, the target altitude is the average of the two "equivalent" estimates for the segment IT. From the dataset $[D_{12}, E_1, A_1, E_2, A_2]$, evaluate the formulation above,

$$H_{target} = 0.5\ D_{12}\,\csc(A_1 + A_2)\ \{(tan(E_1)\ sin(A_2)) \ + \ (tan(E_2)\ sin(A_1))\}\ . \tag{9}$$

As either of the interior angles A_1 or A_2 exceed 90 degrees, the corresponding sine term will be shrinking, but still giving two positive terms of about equal magnitude. The measurement is to be discarded whenever the contributions differ by more than 10 %. In Stine[2] you will find the extra computation step of first subtracting the sum of A_1 and A_2 from 180 and then calculating the sine. Since $sin(x) \equiv sin(\pi - x)$, that step is not required, so this formulation, while equivalent, appears a bit different. The tabulated trigonometric function cosecant(x), $csc(x) \equiv 1/sin(x)$, has been used in this final formulation

In any practical field measurement situation all that is necessary is to first measure your baseline, and then have handy a set of tables for the factor arising from the ground plane triangle that provides A_1 and A_2. A convenient way to represent that table is to start with the sum, which must lay between 0 and 180 degrees. If the sum is near zero or 180, then the image point (I) is very close to the baseline on one side or the other. If the sum is near 90 degrees, then the image point is far from the baseline. Higher precision in measuring the angles is required as they approach zero, for the ratio we want is then a quotient of smaller and smaller numbers. For this reason it is always recommended, and usually possible, to move the baseline off to the side of the launch range, and to situate it more or less along the probable direction of travel for your targets. Similar constraints apply to the elevation angles. If the target gets nearly overhead with respect to either one of the observers, then the tangent of the corresponding elevation angle becomes arbitrarily large — small errors in the measurement produce large swings in the tangent. Hence the best policy is to keep the baseline "out from under the rockets"!

So what is the best structure for a table of ground plane triangle factors? The tables shown below are constructed to take any pair of azimuth angles [A_1,A_2] and show the value of $\mathcal{S} \equiv csc(A_1 + A_2)$. Since the sum of these angles must be less than 180, the table entries are zero for those combinations greater than this value. Moreover, since only one of the azimuth angles can exceed 90, the tables are broken up for those two cases. Simply choose the second table if only one measured azimuth is acute, but use the first if both are acute.

Distance Factor For Both Azimuth Angles Acute

A [deg]	......	2.00000	5.00000	15.00000	25.00000	35.00000	45.00000	55.00000	65.00000	75.00000	85.00000	87.00000
....	sin(A)	.03490	.08716	.25882	.42262	.57358	.70711	.81915	.90631	.96593	.99619	.99863
2.0	.03490	14.33559	8.20551	3.42030	2.20269	1.66164	1.36733	1.19236	1.08636	1.02630	1.00137	1.00015
5.0	.08716	8.20551	5.75877	2.92380	2.00000	1.55572	1.30541	1.15470	1.06418	1.01543	1.00000	1.00061
15.0	.25882	3.42030	2.92380	2.00000	1.55572	1.30541	1.15470	1.06418	1.01543	1.00000	1.01543	1.02234
25.0	.42262	2.20269	2.00000	1.55572	1.30541	1.15470	1.06418	1.01543	1.00000	1.01543	1.06418	1.07853
35.0	.57358	1.66164	1.55572	1.30541	1.15470	1.06418	1.01543	1.00000	1.01543	1.06418	1.15470	1.17918
45.0	.70711	1.36733	1.30541	1.15470	1.06418	1.01543	1.00000	1.01543	1.06418	1.15470	1.30541	1.34563
55.0	.81915	1.19236	1.15470	1.06418	1.01543	1.00000	1.01543	1.06418	1.15470	1.30541	1.55572	1.62427
65.0	.90631	1.08636	1.06418	1.01543	1.00000	1.01543	1.06418	1.15470	1.30541	1.55572	2.00000	2.13005
75.0	.96593	1.02630	1.01543	1.00000	1.01543	1.06418	1.15470	1.30541	1.55572	2.00000	2.92380	3.23607
85.0	.99619	1.00137	1.00000	1.01543	1.06418	1.15470	1.30541	1.55572	2.00000	2.92380	5.75877	7.18530
87.0	.99863	1.00015	1.00061	1.02234	1.07853	1.17918	1.34563	1.62427	2.13005	3.23607	7.18530	9.56677

Distance Factor For One Azimuth Angle Acute

A [deg]	......	90.0000	92.0000	95.0000	105.0000	115.0000	125.0000	135.0000	145.0000	155.0000	165.0000	175.0000	177.0000
....	sin(A)	1.00000	.99939	.99619	.96593	.90631	.81915	.70711	.57358	.42262	.25882	.08716	.05234
2.0	.03490	1.00061	1.00244	1.00751	1.04569	1.12233	1.25214	1.46628	1.83608	2.55930	4.44541	19.10732	57.29869
5.0	.08716	1.00382	1.00751	1.01543	1.06418	1.15470	1.30541	1.55572	2.00000	2.92380	5.75877	.00000	.00000
15.0	.25882	1.03528	1.04569	1.06418	1.15470	1.30541	1.55572	2.00000	2.92380	5.75877	.00000	.00000	.00000
25.0	.42262	1.10338	1.12233	1.15470	1.30541	1.55572	2.00000	2.92380	5.75877	.00000	.00000	.00000	.00000
35.0	.57358	1.22077	1.25214	1.30541	1.55572	2.00000	2.92380	5.75877	.00000	.00000	.00000	.00000	.00000
45.0	.70711	1.41421	1.46628	1.55572	2.00000	2.92380	5.75877	.00000	.00000	.00000	.00000	.00000	.00000
55.0	.81915	1.74345	1.83608	2.00000	2.92380	5.75877	.00000	.00000	.00000	.00000	.00000	.00000	.00000
65.0	.90631	2.36620	2.55930	2.92380	5.75877	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
75.0	.96593	3.86370	4.44541	5.75877	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
85.0	.99619	11.47371	19.10732	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
87.0	.99863	19.10732	57.29869	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

WORKSHEET

1. Note the $\mathcal{S}$ value for the azimuth set [A_1,A_2]:$\mathcal{S} =$.
2. Note the sine of each azimuth: $sin(A_1) =$, $sin(A_2) =$.
3. Compute the elevation tangents: $tan(E_1) =$, $tan(E_2) =$.
4. Compute the factor $0.5\ D_{12}\ \mathcal{S}$
5. Compute the factors $sin(A_1)\,tan(E_2) =$, $sin(A_2)\,tan(E_1) =$, which should be within 10 % of each other.
6. Compute the value for H_{target} from equation 9:

3 Development of a Three Observer Altitude Rule

It is possible of course to use the two observer rule in a pairwise sense when one has three observation stations, and therefore three datasets: [D_{12},E_1,A_1,E_2,A_2], [D_{13}, E_1,B_1,E_3,B_3], and [D_{23},E_2,C_2, E_3,C_3]. Here we just evaluate the worksheet given in last section for each combination and average. There is an automatic means to reject bad data that may occur if the target is nearly overhead for any the observation points, and the clear potential for a general improvement in accuracy as well as reliability.

What can be developed for three observers that is in some way better than this? Well, in the special case that all observers are in a straight line along the ground, one can show that **only elevation angles** need be measured. Insofar as quick sighting for two angles per observer is a bit more complicated than sighting for only one, the existence of such a scheme can reduce the altitude computation to a simpler and more precise set of operations.

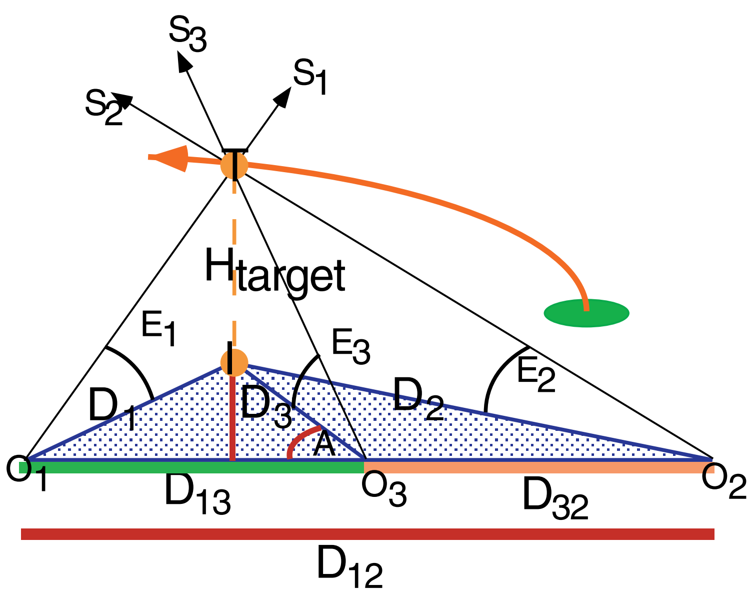


Fig. 3 Elevation Only Geometry

3.1 Proof for Three Colinear Observers

So, once again, back to the “theory zone” whence to consider the geometry of Fig. 3: each of three observers measures the elevation angle of a target. The baseline distances are shown in the figure, so that just as above, we have five measurements in the dataset: [D_{13},D_{32},E_1,E_2,E_3]. The vertex angle associated with the triangles in the ground plane including O_3 is A. The distances from the observers to the ground plane image point (I) are D_1,D_2,and D_3.

In our previous derivation, a common segment (IJ) normal to the baseline was used to partition that baseline. In like manner we can apply the Law of Cosines to the two ground plane triangles: [O_1 O_3 I] and [O_2 O_3 I] to obtain two expressions relating the various distances in these triangles.

$$D_1^2 = D_3^2 + D_{13}^2 - 2\ D_{13} D_3\ cos(A)\ , D_2^2 = D_3^2 + D_{32}^2 + 2\ D_{32} D_3\ cos(A)\ . \quad (10)$$

Notice that the relation for D_2 carries a positive sign for the cosine term, arising from the fact: cos (π - A) = - cos (A) in the triangle [O_2 O_3 I]. Now we may eliminate the common factor $2D_3cos(A)$ between these relations, and with it the last explicit reference to any azimuth in the ground plane:

$$\frac{D_2^2 - D_3^2 - D_{32}^2}{D_{32}} = 2D_3cos(A) = \frac{D_3^2 - D_1^2 + D_{13}^2}{D_{13}}\ . \quad (11)$$

Next, divide out the factors containing the baseline distances, and arrange the simple baseline terms on the right to obtain:

$$\frac{D_2^2 - D_3^2}{D_{32}} + \frac{D_1^2 - D_3^2}{D_{13}} = D_{13} + D_{32}\ . \quad (12)$$

Within equation 12, eliminate the ground plane to image distances by making the following substitutions:

$$D_1 = H\ cot(E_1)\ ; D_2 = H\ cot(E_2)\ ; D_3 = H\ cot(E_3)\ , \quad (13)$$

arising from the common altitude segment H and our three elevations. Finally, make the following substitutions partioning the baseline by means of a factor δ which is always between 0 and 1,

$$D_{13} = \delta D_{12}\ ; D_{23} = (1-\delta) D_{12}\ ; D_{12} = D_{13} + D_{32}\ . \quad (14)$$

Upon collecting terms and extracting the square root of the trigonometric function expression, we obtain:

$$H_{target} = \frac{D_{12}}{\left\{ \left(\frac{cot^2(E_1)\ -\ cot^2(E_3)}{\delta} \right)\ +\ \left(\frac{cot^2(E_2)\ -\ cot^2(E_3)}{1-\delta} \right) \right\}^{1/2}}\ . \quad (15)$$

Stine[3] claims incorrectly that such a formulation is only possible for the case of equidistant observers, viz. δ=0.5. Notice that a proper set of observations will **always** retain a positive definite cotangent term in the denominator because the largest elevation angle is always the one corresponding to the observer nearest the ground plane image of the target, usually the middle observer. If the "cotangant denominator" term ever evaluates negative, the obervations are flawed and incompatible. The measurement must be disregarded.

3.2 Application of the Three Observer Rule

For the rule just derived, the beautiful advantage is the recording of three angles rather than four. Moreover, since we only measure elevations, simpler pointing and tracking equipment can be used. The central observer can be spaced at any interval along the baseline, we need only record the fixed fraction δ at the start of tracking when we measure the baseline. Since only the cotangent function is needed there are no tables to build, a standard hand calculator will easily suffice.

In the (rare) case that the target goes directly over one of the observers, that corresponding cotangent factor simply vanishes. The problem is then a solution for a plane right triangle with either of the two baseline fragments as the base. The altitude can then be computed directly from the elevation cotangent from either triangle, viz. equation 13 with D_j equal to δD_{12} for j= 1, 2, or 3.

WORKSHEET

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1. Note the baseline distance value:$D_{12} =$ ______ .
2. Note the δ value for the baseline:$\delta =$ ______ .
3. Compute the elevation cotangents: $cot(E_1) =$ ______ , $cot(E_2) =$ ______ , $cot(E_3) =$ ______ .
4. Compute the value for H_{target} from equation 15:

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4 Advanced Topics and Problems

The results just developed are readily extended in several directions that may also be of practical use. It is easy to see that once the altitude is determined by any of these techniques, the complete expression for the range to the target from any observer station can be stated. Develop range formulas for one or more of the altitude relations we have constructed.

Similar considerations occur in radio direction finding (DF), although we almost never have precise pointing accuracy in the emergency location transmitter (ELT) signals. So first, examine the range formulations that you derived above to determine the sensitivity of the final estimate to errors in the pointing. Do this by substituting (E+δE) for an elevation (and A+δA for an azimuth) and determined how the final range changes for a small change δE or δA.

Take the range formula that is least sensitive to the pointing error and imagine "turning the original problem on its side". By this I mean we take three or more direction measurements from the air to a ground target. The range is the direct distance along the line of sight, but the length we care about for DF work is the distance along the ground. It is that distance which now corresponds to the altitude we calculated for the rockets. Develop formulations which show a ground team an inferred ELT location (range and azimuth) from three or four direction measurements in an airplane assumed to fly at fixed altitude on a straight course directly over your location, which plays a role similar to the "ground plane image" point of the altitude calculation. How much is the accuracy of this estimate improved if the airplane now flies a second pass on a course perpendicular to the first and reports a similar set of direction measurements?

References

[1] G. Harry Stine, *Handbook of Model Rocketry*, 6^{th} ed., Wiley, NY 1994

[2] ibid, p.277 – 282

[3] ibid, p.282